Soft Computing Fusion with Applications

www.scfa.reapress.com

Soft. Comput. Fusion. Appl. Vol. 2, No. 3 (2025) 127–133.

Paper Type: Original Article

Algorithms for Clustering Fuzzy Soft Sets Based on Their Energies

Nenad Stojanović[1,*], Aslıhan Sezgin[2]

[1] Faculty of Science, University of Kragujevac, Radoja Domanovica 12, Kragujevac, 34000, Serbia; nenad.stojanovic@pmf.kg.ac.rs.

[2] Department of Mathematics and Science Education, Faculty of Education, Amasya University, Amasya, Turkey; aslihan.sezgin@amasya.edu.tr.

Citation:

Received: 23 December 2024 *Revised: 13 March 2025* *Accepted: 09 May 2025*	Stojanović, N., & Sezgin, A. (2025). Algorithms for clustering fuzzy soft sets based on their energies. *Soft Computing Fusion with Applications*, 2(3), 127-133.

Abstract

In this paper, we continue the study of fuzzy soft sets and their applications. Besides the significance of energy and λ-energy of fuzzy soft sets for developing decision-making algorithms, these energies are also crucial for forming data clustering algorithms. The main result of this work is the development of data clustering algorithms based on the energies of fuzzy soft sets.

Keywords: Fuzzy soft set, Energy, Singular values.

1 | Introduction

Information encountered daily in real environments is filled with uncertainties, imprecision, and ambiguities. Every scientific discipline requires precisely defined data; otherwise, accurate reasoning is not possible. A scientific discipline is the main reason researchers are increasingly interested in uncertainty modeling.

Standard mathematical tools often fail to solve such problems. Molodtsov [1] introduced soft set theory as a new mathematical tool for dealing with uncertainties and imprecision. According to Molodtsov [1], soft set theory has been very successful in various mathematical areas, including operations research, game theory, measure theory, and most importantly, machine learning. Soon after the introduction of the soft set concept, many operations on soft sets were defined [2], [7], all aimed at improving decision-making methods based on soft sets. However, despite the wide application of soft set theory, many researchers have enhanced the theory by introducing the concept of fuzzy soft sets [8].

Justifiably, there is a wealth of literature on fuzzy soft sets and their applications, including many generalizations and their limitations. Order ranking and choice value are two different approaches applied in

Corresponding Author: nenad.stojanovic@pmf.kg.ac.rs

https://doi.org/10.22105/scfa.vi.34

soft set theory and fuzzy soft set theory to decision-making problems. Works in which researchers have examined decision-making problems contain various proposed algorithms [9], [14].

On the other hand, graph theory is one of the fundamental theories and mathematical tools in various disciplines, including various areas of computer science and chemistry. Graph energy is a crucial concept currently being studied by many researchers. The concept of graph energy was introduced by Gutman in 1978 in his work [15] as the sum of the absolute values of the graph's eigenvalues. Graph energy is currently intensively studied [16], [20]. The basic concept in the study of graph energy is the investigation of matrices and their properties, especially eigenvalues and singular values. In the work [13], the concept of fuzzy soft set energy was defined, and an algorithm for efficient decision-making was presented.

In Section 2 of our paper, the basic concepts of soft set theory and fuzzy soft set theory, as well as their energies, are outlined. Section 3 presents the main part of our work, introducing an innovative method for clustering fuzzy soft sets based on the energies of fuzzy soft sets introduced in the work [13].

2 | Preliminaries

In this section, the basic concepts of fuzzy soft sets, as well as the numerical characteristics that characterize fuzzy soft sets, called energies of fuzzy soft sets, will be presented [2], [3], [10], [13], [21].

To begin, as done in [21], defining a soft set requires us to consider a universe, denoted as U, which is the set upon which we build the soft set. Next, we consider a set E, which represents the set of parameters. As is well-known, we denote the power set of U as $P(U)$. Let $A \subset E$.

Definition 1 ([21]). A soft set, denoted as F_A, over the universe U, is a set defined by the mapping f_A such that $f_A\colon E \to P(U)$, where $f_A(x) = \emptyset$ if $x \notin A$.

As is customary, the mapping f_A is called the approximate function of the soft set for each $x \in E$. We can also represent the soft set F_A over the universe U using ordered pairs. Such a representation is more intuitive, as the soft set F_A can be written as

$$F_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\}.$$

The set of all soft sets over the universe U is commonly denoted as $S(U)$.

Definition 2 ([22]). A fuzzy set X over the universe U is defined by a function μ_X that represents the mapping $\mu_X : U \to [0, 1]$, where μ_X is called the membership function of X, and the value $\mu_X(u)$ is referred to as the degree of membership of an element $u \in U$ in the fuzzy set X.

Hence, a fuzzy set X over U can be represented as follows: $X = \{(\mu_X(u)/u) \mid u \in U, \mu_X(u) \in [0,1]\}$. The set of all fuzzy sets over the universe U is commonly denoted as $F(U)$.

Next, we define fuzzy soft sets, similar to how it was done in [10].

Definition 3 ([10]). A fuzzy soft set, denoted as Γ_A, over the universe U is a set defined by the function γ_A, representing a mapping $\gamma_A : E \to F(U)$, such that $\gamma_A(x) = \emptyset$ if $x \notin A$.

Similar to fuzzy sets, γ_A is called the fuzzy approximate function of the fuzzy soft set Γ_A, and the value $\gamma_A(x)$ is a set called the x-element of the fuzzy soft set for all $x \in E$. Therefore, a fuzzy soft set Γ_A over the universe, U can be represented as a set of ordered pairs in the following way

$$\Gamma_A = \{(x, \gamma_A(x)) \mid x \in E, \gamma_A(x) \in F(U)\}.$$

The set of all fuzzy soft sets over the universe U is commonly denoted as $FS(U)$.

In the following, we will use the following notations: $\Gamma_A, \Gamma_B, \Gamma_C, \ldots$ for fuzzy soft sets, and $\gamma_A, \gamma_B, \gamma_C, \ldots$ for their fuzzy approximate functions, respectively.

There are also fuzzy soft sets with special characteristics, and for that reason, they have specific, distinct names.

Definition 4 ([10]). Let Γ_A be a fuzzy soft set. If $\gamma_A(x) = \emptyset$ for all $x \in E$, then the fuzzy soft set Γ_A is called an empty fuzzy soft set, denoted as Γ_Φ.

Definition 5 ([10]). Let Γ_A be a fuzzy soft set. If $\gamma_A(x) = U$ for all $x \in E$, then the fuzzy soft set Γ_A is called an A-universal fuzzy soft set, denoted as $\Gamma_{\tilde{A}}$.

We can compare two fuzzy soft sets as follows.

Definition 6 ([10]). Let Γ_A and Γ_B be fuzzy soft sets. The fuzzy soft set Γ_A is a fuzzy soft subset of Γ_B, denoted as $\Gamma_A \tilde{\subseteq} \Gamma_B$, if $\gamma_A(x) \subseteq \gamma_B(x)$ for all $x \in E$.

Definition 7 ([10]). Let Γ_A and Γ_B be fuzzy soft sets. The fuzzy soft sets Γ_A and Γ_B are fuzzy soft equivalent, denoted as $\Gamma_A = \Gamma_B$, if and only if $\gamma_A(x) = \gamma_B(x)$ for all $x \in E$.

Similarly to operations with classic sets, we can consider operations with fuzzy soft sets.

Definition 8 ([10]). Fuzzy soft set $\Gamma_A^{\tilde{c}}$ is the complement of fuzzy soft set Γ_A, such that $\gamma_{A^{\tilde{c}}}(x) = \gamma_A^c(x)$ for all $x \in E$, where $\gamma_A^c(x)$ is the complement of the set $\gamma_A(x)$.

Definition 9 ([10]). The union of fuzzy soft sets Γ_A and Γ_B, denoted as $\Gamma_A \tilde{\cup} \Gamma_B$, is defined by the fuzzy approximative function $\gamma_{A\tilde{\cup}B}(x) = \gamma_A(x) \cup \gamma_B(x)$ for all $x \in E$.

Definition 10 ([10]). The intersection of fuzzy soft sets Γ_A and Γ_B, denoted as $\Gamma_A \tilde{\cap} \Gamma_B$, is defined by the fuzzy approximative function $\gamma_{A\tilde{\cap}B}(x) = \gamma_A(x) \cap \gamma_B(x)$ for all $x \in E$.

The properties of the mentioned operations and relations defined with fuzzy soft sets can be found in [10], and you can learn more about soft sets and fuzzy sets, as well as their operations, in [2], [3], [21], [22].

In the work [13], an innovative representation of fuzzy soft sets using well-known linear algebra concepts [23] was introduced, i.e., the fuzzy soft set is uniquely expressed in the form of a rectangular matrix.

Definition 11 ([13]). Let Γ_A be a fuzzy soft set. Suppose $U = \{u_1, u_2, \ldots, u_m\}$, $E = \{x_1, x_2, \ldots, x_n\}$ and $A \subseteq E$. Then the fuzzy soft set Γ_A can be represented by the *Table 1*.

Table 1. The fuzzy soft set Γ_A.

Γ_A	x_1	x_2	…	x_n
u_1	$\mu_{\gamma_A(x_1)}(u_1)$	$\mu_{\gamma_A(x_2)}(u_1)$	…	$\mu_{\gamma_A(x_n)}(u_1)$
u_2	$\mu_{\gamma_A(x_1)}(u_2)$	$\mu_{\gamma_A(x_2)}(u_2)$	…	$\mu_{\gamma_A(x_n)}(u_2)$
⋮	⋮	⋮	⋱	⋮
u_m	$\mu_{\gamma_A(x_1)}(u_m)$	$\mu_{\gamma_A(x_2)}(u_m)$	…	$\mu_{\gamma_A(x_n)}(u_m)$

where $\mu_{\gamma_A(x)}$ is the membership function of γ_A.

If $a_{ij} = \mu_{\gamma_A(x_j)}(u_i)$ for every $i = 1,2,\ldots,m$ and every $j = 1,2\ldots,n$, then the fuzzy soft set Γ_A is uniquely characterized by the matrix

$$[a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and is referred to as the fuzzy soft matrix of the fuzzy soft set Γ_A over the universe U, of size $m \times n$.

Using the knowledge of linear algebra, especially eigenvalues and singular values of the matrix, similar to graph theory, the concept of energy was defined in the work [13].

Definition 12 ([13]). The energy of a fuzzy soft set Γ_A, denoted as $E(\Gamma_A)$, is defined as $E(\Gamma_A) = \sum_{i=1}^{m} \sigma_i$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$ are the singular values of the matrix A corresponding to the fuzzy soft set Γ_A.

Definition 13 ([13]). The λ-energy of a fuzzy soft set Γ_A, denoted as $LE(\Gamma_A)$ is defined as $LE(\Gamma_A) = \sum_{i=1}^{m} \sigma_i^2$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$ are the singular values of the matrix A corresponding to the fuzzy soft set Γ_A.

Since singular values that are equal to 0 do not affect the value of energy and λ-energy of the fuzzy soft set Γ_A, we will assume that singular values are positive. Knowing the basic properties of matrices and eigenvalues, as well as singular values, the λ-energy of the fuzzy soft set can be determined in multiple ways, as follows:

$$LE(\Gamma_A) = \sigma_1^2(A) + \sigma_2^2(A) + \cdots + \sigma_m^2(A) = tr\left(A \cdot A^T\right),$$

$$LE(\Gamma_A) = tr\left(A \cdot A^T\right) = \sum_{i,j} |a_{ij}|^2 = \lambda_1(A) + \lambda_2(A) + \cdots + \lambda_m(A),$$

where $\lambda_1(A),\ \lambda_2(A), \dots, \lambda_m(A)$ are the eigenvalues of the square matrix $A \cdot A^T$.

In Section 3, we continue to study fuzzy soft sets in terms of their applications.

3 | Fuzzy Soft Data Clustering

The generally accepted stance is that distances and similarity coefficients play a crucial role in data clustering. Clustering can also be performed based on the distance energies (λ-energies) of fuzzy soft sets introduced in the work [13]. Clustering algorithms are based on the distance between clusters, where each data point (Fuzzy soft set) is initially placed in its own cluster, and then, by calculating the distance between clusters, the closest clusters are merged at each step, continuing until we reach a single cluster. Let's assume we have k fuzzy soft sets $(\Gamma_A)_1, (\Gamma_A)_2, \dots, (\Gamma_A)_k$, all defined on the universal set U. The clustering algorithms are given by the following steps (The symbol C_i is used to represent cluster i).

Step 1. Consider each fuzzy soft set as one cluster, i.e., let

$$C_1 = \{(\Gamma_A)_1\}, C_2 = \{(\Gamma_A)_2\}, \dots, C_k = \{(\Gamma_A)_k\}.$$

Step 2. Using the classical Euclidean distance d on the real line, determine the distance between clusters C_i and C_j using the formula

$$D\left(C_i, C_j\right) = \min\left\{d(E(a), E(b)) \mid a \in C_i, b \in C_j\right\}.$$

Step 3. Integrate the two closest clusters (i.e., the clusters with the smallest distance).

Step 4. After integrating clusters, recompute the distance values between the newly obtained clusters.

Step 5. Repeat *Steps 3* and *4* until we reach a single cluster.

The mentioned algorithm is an energy-based clustering algorithm for fuzzy soft sets. However, the clustering algorithm based on λ-energy differs from the mentioned algorithm only in *Step 3*. Specifically, for calculating the distance between clusters, we use λ-energy, i.e., we use the formula

$$D\left(C_i, C_j\right) = \min\left\{d(LE(a), LE(b)) \mid a \in C_i, b \in C_j\right\}.$$

To illustrate the application of the formulated algorithms, let's consider the following real-world example.

Example 1. The example illustrates clustering of the performance of five different energy projects evaluated by some experts. Based on the collected data, fuzzy soft sets $(\Gamma_A)_1$, $(\Gamma_A)_2$, $(\Gamma_A)_3$, $(\Gamma_A)_4$ and $(\Gamma_A)_5$ are formed, given by *Tables 2-6*:

Table 2. Fuzzy soft set $(\Gamma_A)_1$ for the project P_1.

$(\Gamma_A)_1$	e_1	e_2	e_3	e_4
x_1	0.7	0.8	0.5	0.75
x_2	0.6	0.65	0.6	0.75

Table 3. Fuzzy soft set $(\Gamma_A)_2$ for the project P_2.

$(\Gamma_A)_2$	e_1	e_2	e_3	e_4
x_1	0.65	0.5	0.4	0.65
x_2	0.45	0.55	0.7	0.5

Table 4. Fuzzy soft set $(\Gamma_A)_3$ for the project P_3.

$(\Gamma_A)_3$	e_1	e_2	e_3	e_4
x_1	0.75	0.65	0.45	0.75
x_2	0.65	0.55	0.5	0.75

Table 5. Fuzzy soft set $(\Gamma_A)_4$ for the project P_4.

$(\Gamma_A)_4$	e_1	e_2	e_3	e_4
x_1	0.65	0.6	0.45	0.65
x_2	0.55	0.45	0.75	0.6

Table 6. Fuzzy soft set $(\Gamma_A)_5$ for the project P_5.

$(\Gamma_A)_5$	e_1	e_2	e_3	e_4
x_1	0.65	0.6	0.3	0.65
x_2	0.55	0.45	0.7	0.55

Based on the presented fuzzy soft sets, we can determine their energies and find that

$$E((\Gamma_A)_1) = 2.0370,$$

$$E((\Gamma_A)_2) = 1.8348,$$

$$E((\Gamma_A)_3) = 1.8993,$$

$$E((\Gamma_A)_4) = 1.9150,$$

$$E((\Gamma_A)_5) = 1.8966.$$

Using the algorithm presented at the beginning of this section, we can cluster the data into the desired number of clusters.

Table 7. Data clustering.

Number of Clusters	Method Based on The Energy of Fuzzy Soft Set
5	$\{(\Gamma_A)_1\}, \{(\Gamma_A)_2\}, \{(\Gamma_A)_3\}, \{(\Gamma_A)_4\}, \{(\Gamma_A)_5\}$
4	$\{(\Gamma_A)_1\}, \{(\Gamma_A)_2\}, \{(\Gamma_A)_3, (\Gamma_A)_5\}, \{(\Gamma_A)_4\}$
3	$\{(\Gamma_A)_1\}, \{(\Gamma_A)_2\}, \{(\Gamma_A)_3, (\Gamma_A)_4, (\Gamma_A)_5\}$
2	$\{(\Gamma_A)_1\}, \{(\Gamma_A)_2, (\Gamma_A)_3, (\Gamma_A)_4, (\Gamma_A)_5\}$
1	$\{(\Gamma_A)_1, (\Gamma_A)_2, (\Gamma_A)_3, (\Gamma_A)_4, (\Gamma_A)_5\}$

Now, we can determine the λ-energies of the observed fuzzy soft sets, and we have

$LE((\Gamma_A)_1) = 3.6475,$

$LE((\Gamma_A)_2) = 2.4999,$

$LE((\Gamma_A)_3) = 3.2875,$

$LE((\Gamma_A)_4) = 2.8350,$

$LE((\Gamma_A)_5) = 2.5926.$

Using the algorithm based on λ-energy presented at the beginning of this section, we can cluster the data into the desired number of clusters.

Table 8. Data clustering.

Number of Clusters	Method Based on the λ-Energy of Fuzzy Soft Set
5	$\{(\Gamma_A)_1\}, \{(\Gamma_A)_2\}, \{(\Gamma_A)_3\}, \{(\Gamma_A)_4\}, \{(\Gamma_A)_5\}$
4	$\{(\Gamma_A)_1\}, \{(\Gamma_A)_3\}, \{(\Gamma_A)_2, (\Gamma_A)_5\}, \{(\Gamma_A)_4\}$
3	$\{(\Gamma_A)_1\}, \{(\Gamma_A)_3\}, \{(\Gamma_A)_2, (\Gamma_A)_4, (\Gamma_A)_5\}$
2	$\{(\Gamma_A)_1, (\Gamma_A)_3\}, \{(\Gamma_A)_2, (\Gamma_A)_4, (\Gamma_A)_5\}$
1	$\{(\Gamma_A)_1, (\Gamma_A)_2, (\Gamma_A)_3, (\Gamma_A)_4, (\Gamma_A)_5\}$

If we look at the energies of the observed fuzzy soft sets representing data about the observed projects, we can see that Project 1 has the highest energy. Project 1 has evenly represented all characteristics in both members of the universal set, as well as an even representation of all different characteristics. Energy as a numerical value depends on the overall system and reflects the interconnectedness of all characteristics and their mutual dependence, which is the advantage of this method.

Another positive characteristic of energy is that we have unique decision-making, i.e., it is very rare for two different fuzzy soft sets to have the same energy. In that case, we can consider the λ-energies of such fuzzy soft sets as an alternative method that gives us good results.

4 | Summary and Conclusion

Each new property or characteristic of fuzzy soft sets has prompted further research and exploration to establish comparisons, produce new generalizations, prove additional properties, or generate applications of that property or characteristic, as is currently happening with fuzzy soft set energy. In this paper, we have succeeded in integrating a characteristic from graph theory into the theory of fuzzy soft sets, in terms of tools and mechanisms for constructing algorithms for data clustering. In line with the above, the main result of the paper is the development of a clustering algorithm based on the energy and λ-energy of fuzzy soft sets as numerical characteristics that play a key role in decision-making problems as well.

Acknowledgments

The work of the author was supported by the Serbian Ministry of Science, Technological Development, and Innovation (Agreement No. 451-03-65/2024-03/200122).

Conflicts of Interest

The authors declare that there is no conflict of interest.

References

[1] Molodtsov, D. (1999). Soft set theory—First results. *Computers & mathematics with applications*, *37*(4), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5

[2] Ali, M. I., Feng, F., Liu, X., Min, W. K., & Shabir, M. (2009). On some new operations in soft set theory. *Computers & mathematics with applications*, *57*(9), 1547–1553. https://doi.org/10.1016/j.camwa.2008.11.009

[3] Maji, P. K., Biswas, R., & Roy, A. R. (2003). Soft set theory. *Computers & mathematics with applications*, *45*(4), 555–562. https://doi.org/10.1016/S0898-1221(03)00016-6

[4] Sezgin, A., Ahmad, S., & Mehmood, A. (2019). A new operation on soft sets: Extended difference of soft sets. *Journal of new theory*, (27), 33–42. https://dergipark.org.tr/en/pub/jnt/issue/43609/535589

[5] Sezgin, A., & Atagün, A. O. (2011). On operations of soft sets. *Computers & mathematics with applications*, *61*(5), 1457–1467. https://doi.org/10.1016/j.camwa.2011.01.018

[6] Stojanović, N., & Joksimović, M. B. (2022). Soft outer measure and soft premeasure. *Filomat*, *36*(6), 2129-2141. http://dx.doi.org/10.2298/FIL2206129S

[7] Stojanović, N. S. (2021). A new operation on soft sets: extended symmetric difference of soft sets. *Vojnotehnički glasnik/military technical courier*, *69*(4), 779–791. https://doi.org/10.5937/vojtehg69-33655

[8] Maji, P. K., Biswas, R., & Roy, A. R. (2001). Fuzzy soft sets. *J. fuzzy math*, *9*(3), 598–602. https://www.sid.ir/paper/633097/en

[9] Çağman, N., Çıtak, F., & Enginoğlu, S. (2010). Fuzzy parameterized fuzzy soft set theory and its applications. *Turkish journal of fuzzy systems*, *1*(1), 21–35. https://www.researchgate.net/publication/229041420

[10] Cagman, N., Enginoglu, S., & Citak, F. (2011). Fuzzy soft set theory and its applications. *Iranian journal of fuzzy systems*, *8*(3), 137–147. https://www.researchgate.net/publication/259895126

[11] Feng, F., Jun, Y. B., Liu, X., & Li, L. (2010). An adjustable approach to fuzzy soft set based decision making. *Journal of computational and applied mathematics*, *234*(1), 10–20. https://doi.org/10.1016/j.cam.2009.11.055

[12] Liu, Z., Qin, K., & Pei, Z. (2017). A method for fuzzy soft sets in decision-making based on an ideal solution. *Symmetry*, *9*(10), 246. https://doi.org/10.3390/sym9100246

[13] Mudrić-Staniškovski, L., Djurovic, L., & Stojanović, N. (2024). Energy of a fuzzy soft set and its application in decision-making. *Iranian journal of fuzzy systems*, *21*(2), 35–49. http://dx.doi.org/10.22111/ijfs.2024.46797.8243

[14] Rezaei, K., & Rezaei, H. (2019). New distance and similarity measures for hesitant fuzzy soft sets. *Iranian journal of fuzzy systems*, *16*(6), 159–176. https://doi.org/10.22111/ijfs.2019.5026

[15] Gutman, I. (1978). The energy of a graph, *103*, 1–22.

[16] Gutman, I., Kiani, D., Mirzakhah, M., & Zhou, B. (2009). On incidence energy of a graph. *Linear algebra and its applications*, *431*(8), 1223–1233. https://doi.org/10.1016/j.laa.2009.04.019

[17] Jooyandeh, R., Kiani, D., & Mirzakhah, M. (2009). Incidence energy of a graph. *Match (Mulheim an der ruhr, germany)*, *62*, 561–572. https://www.researchgate.net/publication/235676733

[18] Nikiforov, V. (2007). The energy of graphs and matrices. *Journal of mathematical analysis and applications*, *326*(2), 1472–1475. https://doi.org/10.1016/j.jmaa.2006.03.072

[19] Nikiforov, V. (2016). Beyond graph energy: Norms of graphs and matrices. *Linear algebra and its applications*, *506*, 82–138. https://doi.org/10.1016/j.laa.2016.05.011

[20] Zhang, J., & Li, J. (2012). New results on the incidence energy of graphs. *MATCH commun. math. comput. chem*, *68*(3), 777–803. https://B2n.ir/fy8122

[21] Çağman, N., & Enginoğlu, S. (2010). Soft set theory and uni–int decision making. *European journal of operational research*, *207*(2), 848–855. https://doi.org/10.1016/j.ejor.2010.05.004

[22] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

[23] Aggarwal, C. C., Aggarwal, L. F., & Lagerstrom-Fife. (2020). *Linear algebra and optimization for machine learning* (Vol. 156). Springer. https://doi.org/10.1007/978-3-030-40344-7