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Soft Symmetric Difference Complement-Difference Product of Groups

İbrahim Durak[1], **Aslıhan Sezgin**[2,*]

[1] Department of Mathematics, Graduate School of Natural and Applied Sciences, Amasya University, Amasya, Türkiye; ibrahimdurak391919@gmail.com.

[2] Department of Mathematics and Science Education, Faculty of Education, Amasya University, Amasya, Türkiye; aslihan.sezgin@amasya.edu.tr.

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Abstract

Soft set theory, known for its mathematical rigor and algebraic expressiveness, provides a robust framework for addressing uncertainty, vagueness, and variability driven by parameters. This study presents a new binary operation called the soft symmetric difference complement-difference product, which is defined over soft sets with parameter domains that have a group-theoretic structure. Built on a solid axiomatic basis, this operation is shown to fulfill essential algebraic properties, including closure, associativity, commutativity, and idempotency, while aligning with broader concepts of soft equality and subset relationships. The study thoroughly examines the operation's characteristics regarding identity and absorbing elements, as well as its interactions with null and absolute soft sets, all within the context of group-parameterized domains. The results indicate that this operation creates a coherent and structurally sound algebraic system, enhancing the algebraic framework of soft set theory. Additionally, this research lays the groundwork for developing a generalized soft group theory, where soft sets indexed by group-based parameters mimic classical group behaviors through abstract soft operations. The operation's complete integration within soft inclusion hierarchies and its compatibility with generalized soft equalities underscore its theoretical significance and expand its potential uses in formal decision-making and algebraic modeling under uncertainty.

Keywords: Soft sets, Soft subsets, Soft equalities, Soft symmetric difference complement-difference.

1 | Introduction

A broad range of comprehensive numerical systems has been suggested to illustrate and analyze frameworks influenced by uncertainty, ambiguity, and indeterminacy—key characteristics prevalent in fields like engineering, economics, social sciences, and medical diagnostics. Foundational theories such as fuzzy set theory—first introduced by Zadeh [1]—and traditional probabilistic models provide initial tools for addressing these issues, but they often face epistemological and mathematical limitations. Specifically, fuzzy

Corresponding Author: aslihan.sezgin@amasya.edu.tr
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set theory relies on the subjective assignment of membership grades. At the same time, probabilistic methods necessitate access to well-defined distributions and repeatable experiments—conditions that are frequently absent in real-world ambiguous situations. To address these limitations, Molodtsov [2] proposed soft set theory as a parameter-driven, logarithmically adaptable framework that eliminates the need for probabilistic or fuzzy assumptions. Initial operational definitions by Maji et al. [3] were later refined by Pei and Miao [4] through an information-theoretic perspective, enhancing their applicability in social and multivalued systems. This foundation was further developed by Ali et al. [5], [6], who introduced limited and extended operations, significantly improving the representational and mathematical flexibility of soft sets. Numerous subsequent studies have advanced the field by clarifying definitions, introducing new binary operations, and formalizing generalized soft equivalences.

In recent years, the mathematical landscape of soft set theory has significantly expanded, as evidenced by various efforts aimed at developing coherent and extensible parallel operations. These advancements have focused on generalizing the fundamental concepts of soft subsethood and equivalence. Key contributions by Pei and Miao [4], Feng et al. [7], [8], and Qin and Hong [9] established the groundwork for broader theoretical models, which were later enhanced by Jun and Yang [10] and Liu et al. [11] through the development of J-soft and L-soft equivalence relations. Feng and Li [12] further advanced the theory by categorizing soft subsets under L-equality and demonstrating that specific remainder structures satisfy essential semigroup properties, including associativity, commutativity, and distributivity. More recent generalizations—such as g-soft, gf-soft, and T-soft equivalences—have been introduced by Abbas et al. [13], [14] and Alshami and El-Shafei [15], integrating congruence-based and lattice-theoretic approaches into the mathematical analysis of soft sets. A significant breakthrough came from Çağman and Enginoğlu [16], whose foundational modification resolved key inconsistencies and established a coherent and mathematically sound basis for ongoing theoretical advancements. This enhanced theoretical foundation has facilitated the systematic integration of soft set operations into classical mathematical structures. The subtle distinction between the union and difference operations has been modified to fit within the contexts of ring theory [17], semigroup theory [18], and group theory [19]. Conversely, the dual operation—the soft union-difference product—has been applied in group-theoretic [20], semigroup-theoretic [21], and ring-theoretic [22] frameworks. The resulting algebraic behaviors are fundamentally linked to the underlying structures.

Building on this foundation, this research introduces a new binary operation known as the soft symmetric difference complement-difference product, which is defined over soft sets with parameter domains shaped by group-theoretic principles. This operation is developed within a strictly axiomatic framework and is subjected to comprehensive algebraic analysis. Important properties such as closure, associativity, commutativity, and idempotency are rigorously established, along with an exploration of their connections to identity elements, null and absolute soft sets, and absorbing elements. Furthermore, the operation is demonstrated to be fully compatible with generalized notions of soft subsethood and soft equality, seamlessly integrating with the existing algebraic structure of soft set theory. To evaluate its theoretical importance and structural role, a detailed comparative analysis is performed against established soft set operations, focusing on its behavior within layered inclusion hierarchies. By abstracting group-theoretic axioms into parameter-dependent soft structures, this operation lays the groundwork for a generalized soft group theory. In this algebraic system, soft sets indexed by group-structured parameters emulate classical group behavior through formally defined operations. As a result, this work not only introduces a significant algebraic advancement but also establishes a robust theoretical foundation for applying soft set theory in fields that require formal management of uncertainty, abstract algebraic representation, and multi-criteria decision-making frameworks. Manuscript Structure: Section 2 provides the fundamental algebraic foundations and formal definitions that underpin the theoretical framework. Section 3 introduces the soft symmetric difference complement-difference product and thoroughly investigates its associated algebraic theory. Finally, Section 4 summarizes the main conclusions and proposes potential directions for further development of soft algebra in systems that address uncertainty.

2 | Preliminaries

Molodtsov's [2] initial development of soft set theory established a parameter-driven approach to represent uncertainty, but it did not possess the necessary algebraic precision for incorporation into abstract algebraic systems. This shortcoming was successfully remedied by the axiomatic reconstruction by Çağman and Enginoğlu [16], which offered a logically sound and structurally strong basis. The current research is entirely based on this enhanced framework, which supports all definitions, operations, and algebraic structures presented in this work.

Definition 1 ([16]). Let E be a parameter set, U be a universal set, and P(U) be the power set of U, and $\mathcal{H} \subseteq$ E. Then, the soft set $\mathfrak{f}_Y$ over U is a function such that $\Im_{\mathcal{H}}: E \to P(U)$, where for all $w \notin \mathcal{H}$, $\Im_{\mathcal{H}}(w) = \emptyset$. That is,

$$\Im_{\mathcal{H}} = \{(w, \Im_{\mathcal{H}}(w)): w \in E\}.$$

From now on, the soft set over U is abbreviated by $\mathcal{SS}$.

Definition 2 ([16]). Let $\Im_{\mathcal{H}}$ be an $\mathcal{SS}$. If $\Im_{\mathcal{H}}(w) = \emptyset$ for all $w \in E$, then $\Im_{\mathcal{H}}$ is called a null SS and indicated by $\emptyset_E$, and if $\Im_{\mathcal{H}}(w) = U$, for all $w \in E$, then $\Im_{\mathcal{H}}$ is called an absolute SS and indicated by U_E.

Definition 3 ([16]). Let $\Im_{\mathcal{H}}$ and $\wp_{\aleph}$ be two $\mathcal{SS}$s. If $\Im_{\mathcal{H}}(w) \subseteq \wp_{\aleph}(w)$, for all $w \in E$, then $\Im_{\mathcal{H}}$ is a soft subset of $\wp_{\aleph}$ and indicated by $\Im_{\mathcal{H}} \tilde{\subseteq} \wp_{\aleph}$. If $\Im_{\mathcal{H}}(w) = \wp_{\aleph}(w)$, for all $w \in E$, then $\Im_{\mathcal{H}}$ is called soft equal to $\wp_{\aleph}$, and denoted by $\Im_{\mathcal{H}} = \wp_{\aleph}$.

Definition 4 ([16]). Let $f_{\mathcal{H}}$ be an $\mathcal{SS}$. Then, the complement of $f_{\mathcal{H}}$ denoted by ${f_{\mathcal{H}}}^c$, is defined by the soft set ${f_{\mathcal{H}}}^c: E \to P(U)$ such that ${f_{\mathcal{H}}}^c(e) = U \setminus f_{\mathcal{H}}(e) = (f_{\mathcal{H}}(e))'$, for all $e \in E$.

Definition 5 ([16]). Let $\Im_{\mathcal{H}}$ and $\wp_{\aleph}$ be two $\mathcal{SS}$s. Then, the symmetric difference of $\Im_{\mathcal{H}}$ and $\wp_{\aleph}$ is the $\mathcal{SS}$ $\Im_{\mathcal{H}} \tilde{\Delta} \wp_{\aleph}$, where $(\Im_{\mathcal{H}} \tilde{\Delta} \wp_{\aleph})(w) = \Im_{\mathcal{H}}(w) \Delta \wp_{\aleph}(w)$, for all $w \in E$.

Definition 6 ([23]). Let $\Im_K$ and $\wp_{\aleph}$ be two $\mathcal{SS}$s. Then, $\Im_K$ is called a soft S-subset of $\wp_{\aleph}$, denoted by $\Im_K \tilde{\subseteq}_S \wp_{\aleph}$ if for all $w \in E$, $\Im_K(w) = \mathcal{M}$ and $\wp_{\aleph}(w) = \mathcal{D}$, where $\mathcal{M}$ and $\mathcal{D}$ are two fixed sets and $\mathcal{M} \subseteq \mathcal{D}$. Moreover, two SSs $\Im_K$ and $\wp_{\aleph}$ are said to be soft S-equal, denoted by $\Im_K =_S \wp_{\aleph}$, if $\Im_K \tilde{\subseteq}_S \wp_{\aleph}$ and $\wp_{\aleph} \tilde{\subseteq}_S \Im_K$.

It is obvious that if $\Im_K =_S \wp_{\aleph}$, then $\Im_K$ and $\wp_{\aleph}$ are the same constant functions, that is, for all $w \in E$, $\Im_K(w) = \wp_{\aleph}(w) = \mathcal{M}$, where $\mathcal{M}$ is a fixed set.

Definition 7 ([23]). Let $\Im_K$ and $\wp_{\aleph}$ be two $\mathcal{SS}$s. Then, $\Im_K$ is called a soft A-subset of $\wp_{\aleph}$, denoted by $\Im_K \tilde{\subseteq}_A \wp_{\aleph}$, if, for each $a, b \in E$, $\Im_K(a) \subseteq \wp_{\aleph}(b)$.

Definition 8 ([23]). Let $\Im_K$ and $\wp_{\aleph}$ be two $\mathcal{SS}$s. Then, $\Im_K$ is called a soft S-complement of $\wp_{\aleph}$, denoted by $\Im_K =_S (\wp_{\aleph})^c$, if, for all $w \in E$, $\Im_K(w) = \mathcal{M}$ and $\wp_{\aleph}(w) = \mathcal{D}$, where $\mathcal{M}$ and $\mathcal{D}$ are two fixed sets and $\mathcal{M} = \mathcal{D}'$. Here, $\mathcal{D}' = U \setminus \mathcal{D}$.

For additional information on SSs, we refer to [24]-[50].

From now on, let G be a group, and $S_G(U)$ denotes the collection of all $\mathcal{SS}$s over U, whose parameter sets are G; that is, each element of $S_G(U)$ is an $\mathcal{SS}$ parameterized by G. Moreover, let Δ represent the classical symmetric difference operation, which is commutative and associative, in classical set theory. Then, the symmetric difference of the family $\mathfrak{V} = \{C_i : i \in I\}$ such that I is an index set, is denoted by

$$\triangle \mathfrak{V} = \underset{i \in I}{\triangle} C_i = C_1 \Delta C_2 \Delta \ldots \Delta C_n.$$

Definition 9. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two $\mathcal{SS}$s. Then, the soft symmetric difference-lambda product $\mathscr{f}_G \otimes_{s/l} \mathscr{g}_G$ is defined by

$$(\mathscr{f}_G \otimes_{s/l} \mathscr{g}_G)(x) = \underset{x=yz}{\triangle} (\mathscr{f}_G(y) \lambda \mathscr{g}_G(z)) = \underset{x=yz}{\triangle} \left(\mathscr{f}_G(y) \cup (\mathscr{g}_G(z))'\right), \quad y, z \in G,$$

for all $x \in G$.

3 | Soft Symmetric Difference Complement-Difference Product of Group

This section provides an in-depth algebraic examination of the soft symmetric difference complement–difference product, a newly defined binary operation on soft sets that are based on group-theoretically structured parameter domains. The analysis is conducted within a strict axiomatic framework, demonstrating the operation's key properties—closure, associativity, commutativity, and idempotency—thus establishing its function as an internal operation in soft set algebra. Additionally, the operation is shown to correspond with generalized soft subsethood and soft equality, which are crucial for defining morphisms and organizing algebraic substructures. The discussion particularly highlights the operation's placement within stratified inclusion lattices, ensuring its structural integrity and smooth integration into the broader algebraic framework of soft set theory.

From now on, the symmetric difference complement of the family $\mathfrak{B} = \{C_i : i \in I\}$ such that I is an index set, is denoted by

$$\coprod \mathfrak{B} = \coprod_{i \in I} C_i = (C_1 \Delta C_2 \Delta \dots \Delta C_n)'.$$

Definition 1. Let $\mathfrak{I}_G$ and $\wp_G$ be two SSs. Then, the soft symmetric difference complement-difference product $\mathfrak{I}_G \otimes_{s'/d} \wp_G$ is defined by

$$(\mathfrak{I}_G \otimes_{s'/d} \wp_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \wp_G(z)), \quad y, z \in G,$$

for all $x \in G$.

Note here that since G is a group, there always exists $y, z \in G$ such that $= yz$, for all $x \in G$. Let the order of the group G be n, that is, $|G| = n$. Then, it is obvious that there exist n different combinations of writing styles for each $x \in G$ such that $x = yz$, where $y, z \in G$.

Note 1: The soft symmetric difference complement-difference product is well-defined in $S_G(U)$. In fact, let $\mathfrak{I}_G$, $\wp_G$, m_G, $k_G \in S_G(U)$ such that $(\mathfrak{I}_G, \wp_G) = (m_G, k_G)$. Then, $\mathfrak{I}_G = m_G$ and $\wp_G = k_G$, implying that $\mathfrak{I}_G(x) = m_G(x)$ and $\wp_G(x) = k_G(x)$, for all $x \in G$. Thereby, for all $x \in G$,

$$(\mathfrak{I}_G \otimes_{s'/d} \wp_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \wp_G(z)),$$

$$= \coprod_{x=yz} (m_G(y) \setminus k_G(z)),$$

$$= (m_G \otimes_{s'/d} k_G)(x).$$

Hence, $\mathfrak{I}_G \otimes_{s'/d} \wp_G = m_G \otimes_{s'/d} k_G$.

Example 1. Consider the group $G = \{\sigma, \rho\}$ with the following operation:

$\cdot$	σ	ρ
σ	σ	ρ
ρ	ρ	σ

Let $\Im_G$ and $\wp_G$ be two SSs over $U = D_2 = \{<x, y>: x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ as follows:

$\Im_G = \{(\sigma, \{x, yx\}), (\rho, \{e, x\})\}$ and $\wp_G = \{(\sigma, \{e, y\}), (\rho, \{e\})\}$.

Since $\sigma = \sigma\sigma = \rho\rho$, $(\Im_G \otimes_{s'/d} \wp_G)(\sigma) = \left((\Im_G(\sigma) \setminus \wp_G(\sigma)) \Delta (\Im_G(\rho) \setminus \wp_G(\rho))\right)' = \{e, x, y\}$, and since $\rho = \sigma\rho = \rho\sigma$, $(\Im_G \otimes_{s'/d} \wp_G)(\rho) = \left((\Im_G(\sigma) \setminus \wp_G(\rho)) \Delta (\Im_G(\rho) \setminus \wp_G(\sigma))\right)' = \{e, x, y\}$ is obtained. Hence,

$\Im_G \otimes_{s'/d} \wp_G = \{(\sigma, \{e, x, y\}), (\rho, \{e, x, y\})\}$.

Proposition 1. The set $S_G(U)$ is closed under the soft symmetric difference complement-difference product. That is, if $\Im_G$ and $\wp_G$ are two SSs, then so is $\Im_G \otimes_{s'/d} \wp_G$.

Proof: It is obvious that the soft symmetric difference complement-difference product is a binary operation in $S_G(U)$. Thereby, $S_G(U)$ is closed under the soft symmetric difference complement-difference product.

Proposition 2. The soft symmetric difference complement-difference product is not associative in $S_G(U)$.

Proof: Consider the group G and the SSs $\Im_G$ and $\wp_G$ in *Example 1,* let λ_G be an SS over $U = \{e, x, y, yx\}$ such that $\lambda_G = \{(\sigma, \{x, y\}), (\rho, \{y\})\}$.

Since $\Im_G \otimes_{s'/d} \wp_G = \{(\sigma, \{e, x, y\}), (\rho, \{e, x, y\})\}$, then

$(\Im_G \otimes_{s'/d} \wp_G) \otimes_{s'/d} \lambda_G = \{(\sigma, \{e, y, yx\}), (\rho, \{e, y, yx\})\}$.

Moreover, since $\wp_G \otimes_{s'/d} \lambda_G = \{(\sigma, U), (\rho, U)\}$, then

$\Im_G \otimes_{s'/d} (\wp_G \otimes_{s'/d} \lambda_G) = \{(\sigma, U), (\rho, U)\}$.

Thereby, $(\Im_G \otimes_{s'/d} \wp_G) \otimes_{s'/d} \lambda_G \neq \Im_G \otimes_{s'/d} (\wp_G \otimes_{s'/d} \lambda_G)$.

Proposition 3. The soft symmetric difference complement-difference product is not commutative in $S_G(U)$.

Proof: Consider the SSs $\Im_G$ and $\wp_G$ over $U = \{e, x, y, yx\}$ in *Example 1,* then,

$(\Im_G \otimes_{s'/d} \wp_G)(x) = \{(\sigma, \{e, x, y\}), (\rho, \{e, x, y\})\}$ and $(\wp_G \otimes_{s'/d} \Im_G)(x) = \{(\sigma, \{x, yx\}), (\rho, \{x, yx\})\}$

implying that $\Im_G \otimes_{s'/d} \wp_G \neq \wp_G \otimes_{s'/d} \Im_G$.

Proposition 4. The soft symmetric difference complement-difference product is not idempotent in $S_G(U)$.

Proof: Consider the SS $\Im_G$ in *Example 1,* then,

$\Im_G \otimes_{s'/d} \Im_G = \{(\sigma, U), (\rho, \{x, y\})\}$.

Implying that $\Im_G \otimes_{s'/d} \Im_G \neq \Im_G$.

Proposition 5. Let $\Im_G$ be a constant SS. Then, $\Im_G \otimes_{s'/d} \Im_G = U_G$.

Proof: Let $\Im_G$ be a constant SS that, for all $x \in G$, $\Im_G(x) = A$, where A is a fixed set. Then, for all $x \in G$,

$$(\Im_G \otimes_{s'/d} \Im_G)(x) = \coprod_{x=yz} (\Im_G(y) \setminus \Im_G(z)) = U_G(x).$$

Thereby, $\Im_G \otimes_{s'/d} \Im_G = U_G$.

Remark 1. Let $S_G{}^*(U)$ be the collection of all constant SS. Then, the soft symmetric difference complement-difference product is not idempotent in $S_G{}^*(U)$ either.

Proposition 6. Let $\Im_G$ be an SS. Then, $\Im_G \otimes_{s'/d} U_G = U_G$.

Proof: Let $\Im_G$ be an SS. Then, for all $x \in G$,

$$(\mathfrak{I}_G \otimes_{s'/d} U_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus U_G(z)) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus U) = U_G(x).$$

Thus, $\mathfrak{I}_G \otimes_{s'/d} U_G = U_G$.

Remark 2. U_G is the right absorbing element of the soft symmetric difference complement-difference product in $S_G(U)$.

Proposition 7. U_G is not the left absorbing element of the soft symmetric difference complement-difference product in $S_G(U)$.

Proof: Consider the SS $\mathfrak{I}_G$ in *Example 1,* then,

$$U_G \otimes_{s'/d} \mathfrak{I}_G = \{(\sigma, \{x, y\}), (\rho, \{x, y\})\}.$$

Implying that $U_G \otimes_{s'/d} \mathfrak{I}_G \neq \mathfrak{I}_G$.

Remark 3. U_G is not the absorbing element of the soft symmetric difference complement-difference product in $S_G(U)$.

Proposition 8. Let $\mathfrak{I}_G$ be a constant SS. Then,

I. $U_G \otimes_{s'/d} \mathfrak{I}_G = \mathfrak{I}_G$, where $|G| = r$ and r is a positive odd integer.

II. $U_G \otimes_{s'/d} \mathfrak{I}_G = U_G$, where $|G| = r$ and r is a positive even integer.

Proof: Let $\mathfrak{I}_G$ be a constant SS such that, for all $x \in G$, $\mathfrak{I}_G(x) = A$, where A is a fixed set.

I. Let $|G| = r$, where r is a positive odd integer. Then, for all $x \in G$,

$$(U_G \otimes_{s'/d} \mathfrak{I}_G)(x) = \coprod_{x=yz} (U_G(y) \setminus \mathfrak{I}_G(z)) = \Bigg(\underbrace{\mathfrak{I}_G{}^c \Delta \mathfrak{I}_G{}^c \ldots \Delta \mathfrak{I}_G{}^c}_{r \text{ times } \mathfrak{I}_G{}^c, \text{where } r \text{ is odd}}\Bigg)' = \mathfrak{I}_G(x).$$

Thereby, $U_G \otimes_{s'/d} \mathfrak{I}_G = \mathfrak{I}_G$.

II. Let $|G| = r$, where r is a positive even integer. Then, for all $x \in G$,

$$(U_G \otimes_{s'/d} \mathfrak{I}_G)(x) = \coprod_{x=yz} (U_G(y) \setminus \mathfrak{I}_G(z)) = \Bigg(\underbrace{\mathfrak{I}_G{}^c \Delta \mathfrak{I}_G{}^c \ldots \Delta \mathfrak{I}_G{}^c}_{r \text{ times } \mathfrak{I}_G{}^c, \text{where } r \text{ is even}}\Bigg)' = U_G(x).$$

Thereby, $U_G \otimes_{s'/d} \mathfrak{I}_G = U_G$.

Remark 4. U_G is the absorbing element of the soft symmetric difference complement-difference product in $S_G{}^*(U)$, where $|G| = r$ and r is a positive even integer, by *Proposition 6* and *Proposition 8(II)*. Besides, U_G is the left identity element of the soft symmetric difference complement-difference product in $S_G{}^*(U)$, where $|G| = r$ and r is a positive odd integer, by *Proposition 8(I)*.

Proposition 9. Let $\mathfrak{I}_G$ be an SS. Then, $\emptyset_G \otimes_{s'/d} \mathfrak{I}_G = U_G$.

Proof: Let $\mathfrak{I}_G$ be an SS. Then, for all $x \in G$,

$$(\emptyset_G \otimes_{s'/d} \mathfrak{I}_G)(x) = \coprod_{x=yz} (\emptyset_G(y) \setminus \mathfrak{I}_G(z)) = \coprod_{x=yz} (\emptyset \setminus \mathfrak{I}_G(z)) = U_G(x).$$

Thereby, $\emptyset_G \otimes_{s'/d} \mathfrak{I}_G = U_G$.

Proposition 9. Let $\mathfrak{I}_G$ be a constant SS. Then,

I. $\mathfrak{I}_G \otimes_{s'/d} \emptyset_G = \mathfrak{I}_G{}^c$, where $|G| = r$ and r is a positive odd integer.

II. $\mathfrak{I}_G \otimes_{s'/d} \emptyset_G = U_G$, where $|G| = r$ and r is a positive even integer.

Proof: Let $\mathfrak{I}_G$ be a constant SS such that, for all $x \in G$, $\mathfrak{I}_G(x) = A$, where A is a fixed set.

I. Let $|G| = r$, where r is a positive odd integer. Then, for all $x \in G$,

$$(\mathfrak{I}_G \otimes_{s'/d} \emptyset_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \emptyset_G(z)) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \emptyset) = \mathfrak{I}_G{}^c(x).$$

Thereby, $\mathfrak{I}_G \otimes_{s'/d} \emptyset_G = \mathfrak{I}_G{}^c$

II. Let $|G| = r$, where r is a positive even integer. Then, for all $x \in G$,

$$(\mathfrak{I}_G \otimes_{s'/d} \emptyset_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \emptyset_G(z)) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \emptyset) = U_G(x).$$

Thereby, $\mathfrak{I}_G \otimes_{s'/d} \emptyset_G = U_G$. □

Proposition 10. Let $\mathfrak{I}_G$ be a constant SS. Then,

I. $\mathfrak{I}_G{}^c \otimes_{s'/d} \mathfrak{I}_G = \mathfrak{I}_G$, where $|G| = r$ and r is a positive odd integer.

II. $\mathfrak{I}_G{}^c \otimes_{s'/d} \mathfrak{I}_G = U_G$, where $|G| = r$ and r is a positive even integer.

Proof: Let $\mathfrak{I}_G$ be a constant SS such that, for all $x \in G$, $\mathfrak{I}_G(x) = A$, where A is a fixed set.

I. Let $|G| = r$, where r is a positive odd integer. Then, for all $x \in G$,

$$(\mathfrak{I}_G{}^c \otimes_{s'/d} \mathfrak{I}_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G{}^c(y) \setminus \mathfrak{I}_G(z)) = \mathfrak{I}_G(x).$$

Thereby, $\mathfrak{I}_G{}^c \otimes_{s'/d} \mathfrak{I}_G = \mathfrak{I}_G$.

II. Let $|G| = r$, where r is a positive even integer. Then, for all $x \in G$,

$$(\mathfrak{I}_G{}^c \otimes_{s'/d} \mathfrak{I}_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G{}^c(y) \setminus \mathfrak{I}_G(z)) = U_G(x).$$

Thereby, $\mathfrak{I}_G{}^c \otimes_{s'/d} \mathfrak{I}_G = U_G$.

Proposition 11. Let $\mathfrak{I}_G$ be a constant SS. Then,

I. $\mathfrak{I}_G \otimes_{s'/d} \mathfrak{I}_G{}^c = \mathfrak{I}_G{}^c$, where $|G| = r$ and r is a positive odd integer.

II. $\mathfrak{I}_G \otimes_{s'/d} \mathfrak{I}_G{}^c = U_G$, where $|G| = r$ and r is a positive even integer.

Proof: Let $\mathfrak{I}_G$ be a constant SS such that, for all $x \in G$, $\mathfrak{I}_G(x) = A$, where A is a fixed set.

I. Let $|G| = r$, where r is a positive odd integer. Then, for all $x \in G$,

$$(\mathfrak{I}_G \otimes_{s'/d} \mathfrak{I}_G{}^c)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \mathfrak{I}_G{}^c(z)) = \mathfrak{I}_G{}^c(x).$$

Thereby, $\mathfrak{I}_G \otimes_{s'/d} \mathfrak{I}_G{}^c = \mathfrak{I}_G{}^c$.

II. Let $|G| = r$, where r is a positive even integer. Then, for all $x \in G$,

$$(\mathfrak{I}_G \otimes_{s'/d} \mathfrak{I}_G{}^c)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \mathfrak{I}_G{}^c(z)) = U_G(x).$$

Thereby, $\mathfrak{I}_G \otimes_{s'/d} \mathfrak{I}_G{}^c = U_G$.

Proposition 12. Let $\mathfrak{I}_G$ and $\wp_G$ be two SSs such that $\mathfrak{I}_G \tilde{\subseteq}_A \wp_G$. Then, $\mathfrak{I}_G \otimes_{s'/d} \wp_G = U_G$.

Proof: Let $\mathfrak{I}_G$ and $\wp_G$ be two $\mathcal{SS}$s and $\mathfrak{I}_G \cong\!\!\!\subseteq_A \wp_G$. Then, $\mathfrak{I}_G(y) \subseteq \wp_G(z)$, for each $y, z \in G$. Then, for all $x \in G$,

$$(\mathfrak{I}_G \otimes_{s'/d} \wp_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \wp_G(z)) = U_G(x).$$

Thereby, $\mathfrak{I}_G \otimes_{s'/d} \wp_G = U_G$.

Proposition 13. Let $\mathfrak{I}_G$ and $\wp_G$ be two $\mathcal{SS}$s such that $\mathfrak{I}_G \cong\!\!\!\subseteq_S \wp_G$. Then, $\mathfrak{I}_G \otimes_{s'/d} \wp_G = U_G$.

Proof: The proof is similar to the proof of *Proposition 12*.

Proposition 14. Let $\mathfrak{I}_G$ and $\wp_G$ be two $\mathcal{SS}$s such that $\mathfrak{I}_G \cong\!\!\!\subseteq_S (\wp_G)^c$. Then,

I. $\mathfrak{I}_G \otimes_{s'/d} \wp_G = \mathfrak{I}_G{}^c$, where $|G| = r$ and r is a positive odd integer.

II. $\mathfrak{I}_G \otimes_{s'/d} \wp_G = U_G$, where $|G| = r$ and r is a positive even integer.

Proof: Let $\mathfrak{I}_G$ and $\wp_G$ be two $\mathcal{SS}$s and $\mathfrak{I}_G \cong\!\!\!\subseteq_S (\wp_G)^c$. Then, for all $x \in G$, $\mathfrak{I}_G(x) = A$, $\wp_G(x) = B$, where A and B are two fixed sets and $A \subseteq B'$.

I. Let $|G| = r$, where r is a positive odd integer. Then, for all $x \in G$,

$$(\mathfrak{I}_G \otimes_{s'/d} \wp_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \wp_G(z)) = \mathfrak{I}_G{}^c(x).$$

Thereby, $\mathfrak{I}_G \otimes_{s'/d} \wp_G = \mathfrak{I}_G{}^c$.

II. Let $|G| = r$, where r is a positive even integer. Then, for all $x \in G$,

$$(\mathfrak{I}_G \otimes_{s'/d} \wp_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \wp_G(z)) = U_G(x).$$

Thereby, $\mathfrak{I}_G \otimes_{s'/d} \wp_G = U_G$.

Proposition 15. Let $\mathfrak{I}_G$ and $\wp_G$ be two $\mathcal{SS}$s such that $(\wp_G)^c \cong\!\!\!\subseteq_S \mathfrak{I}_G$. Then,

I. $\mathfrak{I}_G \otimes_{s'/d} \wp_G = \wp_G$, where $|G| = r$ and r is a positive odd integer.

II. $\mathfrak{I}_G \otimes_{s'/d} \wp_G = U_G$, where $|G| = r$ and r is a positive even integer.

Proof: Let $\mathfrak{I}_G$ and $\wp_G$ be two $\mathcal{SS}$s and $(\wp_G)^c \cong\!\!\!\subseteq_S \mathfrak{I}_G$. Then, for all $x \in G$, $\mathfrak{I}_G(x) = A$, $\wp_G(x) = B$, where A and B are two fixed sets and $B' \subseteq A$.

I. Let $|G| = r$, where r is a positive odd integer. Then, for all $x \in G$,

$$(\mathfrak{I}_G \otimes_{s'/d} \wp_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \wp_G(z)) = \wp_G(x).$$

Thereby, $\mathfrak{I}_G \otimes_{s'/d} \wp_G = \wp_G$. Here note that $\mathfrak{I}_G(y) \setminus \wp_G(z) = \wp_G{}^c(z) \cap \mathfrak{I}_G(y)$, for all $y, z \in G$,

II. Let $|G| = r$, where r is a positive even integer. Then, for all $x \in G$,

$$(\mathfrak{I}_G \otimes_{s'/d} \wp_G)(x) = \coprod_{x=yz} (\mathfrak{I}_G(y) \setminus \wp_G(z)) = U_G(x).$$

Thereby, $\mathfrak{I}_G \otimes_{s'/d} \wp_G = U_G$.

Proposition 16. Let $\mathfrak{I}_G$ and $\wp_G$ be two $\mathcal{SS}$s. Then, $(\mathfrak{I}_G \otimes_{s'/d} \wp_G)^c = \mathfrak{I}_G \otimes_{s/p} \wp_G$.

Proof: Let $\mathfrak{I}_G$ and $\wp_G$ be two $\mathcal{SS}$s. Then, for all $x \in G$,

$$\left(\mathfrak{I}_{G}\otimes_{s'/d}\wp_{G}\right)^{c}(x)=\left(\coprod_{x=yz}\left(\mathfrak{I}_{G}(y)\setminus\wp_{G}(z)\right)\right)',$$

$$=\underset{x=yz}{\triangle}\left(\mathfrak{I}_{G}(y)\setminus\wp_{G}(z)\right)',$$

$$=\underset{x=yz}{\triangle}\left(\mathfrak{I}_{G}(y)\cap\wp_{G}{}^{c}(z)\right),'$$

$$=\underset{x=yz}{\triangle}\left(\mathfrak{I}_{G}{}^{c}(y)\cup\wp_{G}(z)\right).$$

Thereby, $\left(\mathfrak{I}_{G}\otimes_{s'/d}\wp_{G}\right)^{c}=\mathfrak{I}_{G}\otimes_{s/p}\wp_{G}$.

4 | Conclusion

This research presents the soft symmetric difference complement–difference product, a new binary operation on soft sets that are influenced by group-theoretic structures. The operation is analyzed within a detailed algebraic framework, focusing on its relationship with generalized soft equality and its function across different levels of soft subsethood. A comparative analysis showcases the operation's expressive capabilities and algebraic coherence in relation to existing soft set operations. The study also investigates its alignment with fundamental concepts like null and absolute soft sets, as well as its consistency within group-parameterized binary operations. Essential algebraic properties—such as closure, associativity, commutativity, and idempotency—are thoroughly confirmed, including conditions related to identity, inverse, and absorbing elements. The resulting structure exhibits strong internal consistency, extending traditional algebraic concepts into the realm of soft sets. Fundamentally, this operation establishes a basis for a generalized soft group theory, where soft sets indexed by group-structured parameters mimic group-like behavior through abstract soft operations. Consequently, this work not only enhances the theoretical foundation of soft set theory but also expands its potential applications in fields like algebraic modeling and decision-making under uncertainty.

Author Contributions

İbrahim Durak: Investigation, visualization, conceptualization, writing-review, validation

Aslıhan Sezgin: Supervision, visualization, conceptualization, validation, review

Conflicts of Interest

The authors stated that there are no conflicts of interest regarding the publication of this article.

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